Target Localization Using Mobile Sensors and a Decentralized and Distributed Variational Estimator

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***Abstract*— This work proposes a distributed and decentralized observer for position estimation (localization) of a moving target being tracked by multiple mobile sensors. The possibly time-varying set of sensors that have the target in their field of view, is used to create an energy-like quantity that depends on errors in the estimated relative positions and relative velocities of the target as measured by the mobile sensors. The relative velocities need not be measured directly, and can be obtained by filtering the observed relative positions. Each sensor then implements a local version of this distributed observer, and shares relative position information with the other sensors that are tracking the target. The observer is in the form of a variational estimator that is obtained by taking an action functional constructed from the energy-like quantity and dissipating this energy. As a result, the observer is shown to be asymptotically stable. Numerical simulations confirm this stability property and indicate robustness of the distributed observer to bounded measurement errors.**

1. INTRODUCTION

This work considers the problem of a set of mobile cooperative sensors that autonomously work together to locate and track a mobile target. Applications are to networks of unmanned vehicles tasked to locate and track targets: such networks could be homogenous (all vehicles of the same type) or heterogenous (networks of different types of unmanned vehicles possibly spanning different domains like ground, maritime, air and space). This requires autonomous state estimation of position and velocity states of the target by fusing information gathered by the network of sensors.

The proposed distributed observer scheme can enhance the autonomy and reliability of such autonomous mobile sensor networks operating in uncertain environments where internal parameters (like mass and inertia properties of sensors) and environmental factors (like disturbance forces and torques). It also does not require knowledge of the internal dynamics of the mobile sensors or the mobile target being tracked. In practice, the dynamics of such mobile sensors (e.g., unmanned aerial vehicles or UAVs) may not be perfectly known, especially when the vehicle is under the action of poorly known forces and moments due to wind, weather, etc. In such situations, deterministic estimation approaches like deterministic observers turn out to be robust to disturbances acting on the system or system model uncertainties [1], [2], [3].

In recent years, we have developed and tested the concept of *variational estimators*, which is based on applying the framework of variational mechanics to state estimation of rigid body systems [4], [5], [6]. This approach was also applied to relative state estimation between rigid body systems (modeling unmanned vehicles) in [7]. Prior work that relates to the variational estimation schemes are the "minimum energy" or "maximum likelihood" recursive filtering schemes first proposed in [8]. The minimum energy filtering schemes were more recently extended and applied to rigid body state estimation in [9], [10], [11]. However, Mortensen's method of minimum energy filtering, although robust to uncertainties in measurements and dynamics, has a significant drawback. It can only be solved in an approximate sense as the nonlinear minimum-energy estimation problem requires solving the Hamilton-Jacobi-Bellman (HJB) equations for the nonlinear maximum likelihood estimation problem. As the HJB equations cannot be solved exactly, they are solved in an approximate sense usually up to the second-order; in other words, they are at best "near optimal". Moreover, as was shown in our prior work, they are not guaranteed to be nonlinearly stable [12]. The lack of stability is an even more significant drawback of minimum-energy filtering schemes when applied to systems with uncertainties in internal dynamics and disturbances. This provides the motivation behind the variational estimation approach, which guarantees nonlinear stability by dissipating a mechanical energy-like quantity created from the state estimation errors. This dissipation in the energy is achieved by introducing a mechanical damping-like term that is linear in the velocity estimation errors and applying the Lagrange-d'Alembert principle from variational mechanics to the total energy [13], [14], [15].

Here we apply the variational estimation approach to the network of mobile sensors tracking a mobile target, assuming that one or more of the sensors are in direct line-of-sight of the target. The positions and velocities of the sensors are assumed to be known across the network. For ease of analysis, the communications model is based on a simple model where each sensor can communicate with other sensors located within a sphere of known radius. As the network itself is mobile, this means that the network topology is not static and so the controls and communications will have to be coupled so that the network graph remains strongly connected for this distributed variational estimation framework to work. However, we do not deal with the controls-communications coupling in this work, and assume that strong connectivity of the network is maintained so that the distributed observer is able to estimate the position and velocity of the target.

The remainder of this paper is organized as follows. Section 2 lays down the basic assumptions and theoretical

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framework for the distributed observer design. Section 3 gives the main result of this paper: the variational estimator for the position and velocity of the target as a distributed observer. This is obtained by applying the Lagrange-d'Alembert principle from variational mechanics to a Lagrangian created from estimation errors that combine measurements from all sensors directly observing the target. In section 4, we develop a discrete-time version of the variational estimator as a distributed observer, and then use this discrete-time version for numerical simulations on a computer. The discrete-time version is obtained by applying the discrete Lagrange-d'Alembert principle [16] to the same Lagrangian used to design the continuous-time variational estimator in section 3. Finally, section 5 provides a summary of the main result and contributions of this paper and possible related future developments.

2. Framework for Distributed Observer Design

Consider a mobile target being tracked by a network of mobile sensors in three dimensional Euclidean space $\mathbb{R}^3$. To track the mobile target, the target's states need to be estimated in real time by the mobile sensor network. We make the following assumptions about the network of sensors: (1) each sensor has knowledge of its own position and velocity in an inertial coordinate frame; (2) each sensor can communicate back and forth with other sensors that are located within a sphere of known radius centered at this sensor; (3) one or more sensors are in direct line-of-sight of the target; and (4) the sensors are controlled such that there is a communications path between any two sensors in the network. The second and fourth assumptions imply that every sensor in the network is within communications range of at least two other sensors, which makes the network graph 2-edge-connected and therefore strongly connected according to Robbins' theorem [17]. For the target tracking objective, a distributed observer is designed for state estimation of the observed target by the networked multi-sensor system.

Let $\mathcal{N} = \{1, \dots, n\}$ denote the index set of mobile sensors in the multi-sensor system, and $i \in \mathcal{N}$ the index number used to label a particular sensor. A concept diagram of this multi-sensor target tracking problem is given in Fig. 1. Let $(b^i, v^i) \in \mathbb{R}^6$ denote states of the ith sensor, where $b^i \in \mathbb{R}^3$ denotes its inertial position vector and $v^i \in \mathbb{R}^3$ denotes its inertial velocity vector. Let $(b^0, v^0) \in \mathbb{R}^6$ denote states of the target object being tracked by the system. The instantaneous relative position between target and ith sensor and between ith and jth sensors $(i, j \in \mathcal{N})$, are denoted

$$b^{0i} = b^i - b^0 \text{ and } b^{ij} = b^j - b^i. \tag{1}$$

Likewise, the instantaneous relative velocity between target and ith sensor and between ith and jth sensors $(i, j \in \mathcal{N})$, are denoted

$$v^{0i} = v^i - v^0 = \dot{b}^{0i} \text{ and } v^{ij} = v^j - v^i = \dot{b}^{ij}. \tag{2}$$

Denote the larger index set of objects (including sensors and target) by $\bar{\mathcal{N}} = \{0\} \cup \mathcal{N}$, and let μ and ν denote indices

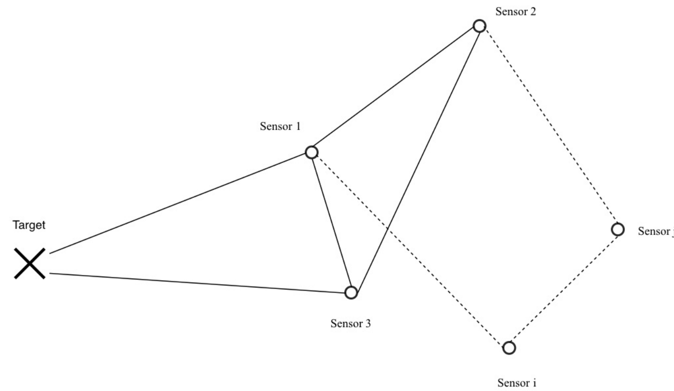

Fig. 1. Inertial relative motion of a target as observed and tracked by a distributed network of mobile sensors.

of a pair of objects, where $\mu, \nu \in \bar{\mathcal{N}}$. Let $b^\mu, b^\nu \in \mathbb{R}^3$ denote the positions of these two objects; therefore $b^{\mu\nu}$ is their relative position as defined by eq. (1). Thus $b^\mu \nu$ is the (instantaneous) relative position of object μ as observed by object ν. The relative velocity between these two objects is $v^{\mu\nu} = \dot{b}^{\mu\nu}$. The relative motion between the sensor i and the target given by (b^{0i}, v^{0i}) can be used for control and tasking of the sensor i, which do not consider here. As estimating the motion of the target is the problem we are considering in this article, we next give the measurement model for these relative motion states.

The quantities b^{0i} and $v^{0i} = \dot{b}^{0i}$ are estimated in continuous time from the observed values, which may have errors due to measurement noise that needs to be filtered. Assuming that these errors can be modeled as additive errors, we denote the corresponding measured quantities as:

$$b^{0i}_m = b^{0i} + \chi^{0i} \text{ and } v^{0i}_m = v^{0i} + v^{0i}, \tag{3}$$

where $\chi^{0i}, v^{0i} \in \mathbb{R}^3$ are the additive measurement noise vectors. The following section gives the deterministic estimator design for these quantities without any assumption on the statistical distribution of these errors.

3. Variational Estimator for Position and Velocity of Target

Let $\mathcal{N}^t_0 \subset \mathcal{N}$ denote the subset of sensors that observe the target at time t; note that this is a time-varying subset of $\mathcal{N}$. Let $i \in \mathcal{N}^t_0$ and let

$$\widehat{b}^{0i} \in \mathbb{R}^3 \text{ and } \widehat{v}^{0i} \in \mathbb{R}^3$$

denote the estimates of b^{0i} and v^{0i} respectively, as obtained from the measurements expressed in (3). As we are interested in directly estimating the absolute position and velocity of the target, we consider the estimates

$$\widehat{b}^0 \in \mathbb{R}^3 \text{ and } \widehat{v}^0 \in \mathbb{R}^3$$

respectively, with measurements from all sensors $i \in \mathcal{N}^t_0$. Therefore the estimation errors in these quantities are:

$$x^0 = \widehat{b}^0 - b^0, \quad u^0 = \widehat{v}^0 - v^0. \tag{4}$$

The variational estimator design described here is a deterministic observer that gives a continuous time observer law for updating the estimates $(\widehat{b}^0, \widehat{v}^0)$ obtained from energy-like quantities defined from these estimation errors.

A. Observer Form of Variational Estimator for Translational Motion of Target

Let λ_i for $i \in \mathcal{N}_0^t$ be positive scalar observer gains that satisfy

$$\lambda_i > 0 \ \text{ s.t. } \sum_{i\in\mathcal{N}_0^t} \lambda_i = 1. \tag{5}$$

The λ_i can be considered as weight factors that place different weights on each of the sensors observing the target based upon knowledge of the accuracy of their measurements. We re-define the state estimation errors in (4) based on directly measured quantities:

$$x^0 = \widehat{b}^0 - \sum_{i\in\mathcal{N}_0^t} \lambda_i (b^i - b^{0i}), \tag{6}$$

$$u^0 = \widehat{v}^0 - \sum_{i\in\mathcal{N}_0^t} \lambda_i (v^i - v^{0i}), \tag{7}$$

which give the target's position estimate and target's velocity estimate errors, respectively. By imposing the condition that $\dot{\widehat{b}}^0 = \widehat{v}^0$, we get

$$\dot{x}^0 = u^0. \tag{8}$$

The energy in these state estimation errors is dissipated by our observer design.

Define a potential energy-like term from the position estimation error given by (6), as follows:

$$\mathcal{U}(x^0) := \frac{1}{2}(x^0)^{\mathrm{T}} K x^0 \ \text{ where } \ K = K^{\mathrm{T}} > 0. \tag{9}$$

This potential-like term is also a measure of the energy in the target's position estimation error based on all sensors $i \in \mathcal{N}_0^t$ that measure the target. A similar kinetic energy-like term can be defined from the velocity estimation error (7), as follows:

$$\mathcal{T}(u^0) := \frac{1}{2}(u^0)^{\mathrm{T}} M u^0 \ \text{ where } \ M = M^{\mathrm{T}} > 0. \tag{10}$$

The positive definite matrices $K, M \in \mathbb{R}^{3\times 3}$ are observer gain matrices that are design parameters. This kinetic energy-like term is a measure of the energy in the target's velocity estimation error from measurements by the sensors $i \in \mathcal{N}_0^t$. Now consider the following Lagrangian constructed from these energy terms in eqs. (9)-(10):

$$\begin{aligned}\mathcal{L}(x^0, u^0) &:= \mathcal{T}(u^0) - \mathcal{U}(x^0) \\ &= \frac{1}{2}(u^0)^{\mathrm{T}} M u^0 - \frac{1}{2}(x^0)^{\mathrm{T}} K x^0.\end{aligned} \tag{11}$$

The observer design that follows is based on this Lagrangian function (11) and a dissipative term that dissipates the total mechanical energy-like quantity given by the sum of $\mathcal{T}(u^0)$ and $\mathcal{U}(x^0)$.

Proposition 3.1: Let $\mathcal{L}(x^0, u^0)$ be the Lagrangian defined by (11) and let $C \in \mathbb{R}^{3\times 3}$ be a positive definite matrix. Then the variational estimator for the translational motion states of the target, obtained by applying the Lagrange-d'Alembert principle to this Lagrangian with the dissipation term $F_D = -Cu^0$, is given by

$$\begin{aligned}\dot{\widehat{b}}^0 &= \widehat{v}^0, \\ \widehat{v}^0 &= \sum_{i\in\mathcal{N}_0^t} \lambda_i (v^i - v^{0i}) + u^0, \\ M\dot{u}^0 &= -Cu^0 - Kx^0,\end{aligned} \tag{12}$$

where x^0 is given by (6) and x^0 and u^0 are related as in (8).

Proof: The first of eqs. (12) is what we set as the relation between the target's position and velocity estimates. The second of eqs. (12) is a re-statement of (7). The third equation is obtained by applying the Lagrange-d'Alembert principle to the Lagrangian (11) with the forcing term (dissipation) F_D. This leads to the following Euler-Lagrange equation with forcing:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}(x^0, u^0)}{\partial u^0} - \frac{\partial \mathcal{L}(x^0, u^0)}{\partial x^0} = F_D. \tag{13}$$

Now substituting the form of the Lagrangian (11) and the dissipation term $F_D = -Cu^0 = -C\dot{x}^0$, we get the third of eqs. (12), which completes this proof. ■

B. Stability of Variational Estimator for Translational Motion of Target

The observer given by Proposition 3.1 is stable as a consequence of the dissipation term introduced. This is shown in the following result.

Theorem 3.2: The variational estimation scheme given by Proposition 3.1 is asymptotically stable and the state estimation errors $(x^0, u^0) \in \mathbb{R}^6$ asymptotically converge to zero.

Proof: The stability of this observer follows from using the total energy-like function:

$$\begin{aligned}\mathcal{E}(x^0, u^0) &:= \mathcal{T}(u^0) + \mathcal{U}(x^0) \\ &= \frac{1}{2}(u^0)^{\mathrm{T}} M u^0 + \frac{1}{2}(x^0)^{\mathrm{T}} K x^0.\end{aligned} \tag{14}$$

as a Lyapunov function, and taking its time derivative. After substituting the observer form given by (12), the time derivative evaluates to:

$$\dot{\mathcal{E}}(x^0, u^0) = -(u^0)^{\mathrm{T}} C u^0 \leq 0. \tag{15}$$

The above time derivative is negative semi-definite in the "error states" (x^0, u^0), from which we conclude stability. Now applying LaSalle's invariance principle in the form given by Theorem 8.4 of [18], we see that the largest invariant subset of the set where $\dot{\mathcal{E}}(x^0, u^0) = 0$ is the singleton set $(x^0, u^0) = (0, 0)$. Therefore, the error states (x^0, u^0) converge to zero asymptotically with time. ■

Note that Theorem 3.2 states that the observer form given by Proposition 3.1 is asymptotically stable in the absence of measurement errors in b^{0i} and v^{0i}. We assume that the positions and velocities of the sensors $i \in \mathcal{N}_0^t$ are perfectly known or estimated onboard these sensors. In the presence of measurement errors in the relative position and relative velocity of the target as given by (3), the effect of the variational estimator will be to integrate the noise in the relative position measurement and attenuate the error in the relative velocity measurements, as given by eqs. (12). However, the strong stability property of this estimator as stated in Theorem 3.2 makes it robust to bounded measurement noise.

In the presence of measurement noise, the variational estimator for the target's position and velocity is given by the following set of equations:

$$\begin{aligned} \dot{\widehat{b}}^0 &= \widehat{v}^0, \\ \widehat{v}^0 &= \sum_{i\in\mathcal{N}_0^t} \lambda_i(v^i - v_m^{0i}) + u^0, \\ M\dot{u}^0 &= -Cu^0 - Kx^0, \end{aligned} \tag{16}$$

The above implementation of the variational estimator can be initialized as follows:

$$\begin{aligned} x^0(t_0) &= \widehat{b}^0(t_0) - \sum_{i\in\mathcal{N}_0^{t_0}} \lambda_i(b^i - b_m^{0i})(t_0), \\ u^0(t_0) &= \widehat{v}^0(t_0) - \sum_{i\in\mathcal{N}_0^{t_0}} \lambda_i(v^i - v_m^{0i})(t_0). \end{aligned} \tag{17}$$

The initial values in eqs. (17) are used in the last of eqs. (16) to update u^0 at later times, and the updated u^0 is used in the first two of eqs. (16) to update $\widehat{b}^0$ and $\widehat{v}^0$.

4. Discrete Variational Estimator and Numerical Simulations

The continuous-time variational estimation scheme given in section 3 is not best suited for implementation on a computer, unlike discrete-time estimation schemes. For use in both numerical simulations as well as onboard computer implementation in hardware, a discrete-time estimation scheme is advantageous. Here we obtain a discrete-time version of the variational estimation scheme outlined in the previous section, and use it for numerical simulations on a computer to verify its performance.

A. Discrete Variational Estimator for Translational Motion of Target

Consider the Lagrangian given by (11) evaluated now in discrete time as follows:

$$\begin{aligned} \mathcal{L}(x_k^0, u_k^0) := \mathcal{L}_k &= \mathcal{T}(u_k^0) - \mathcal{U}(x_k^0) \\ &= \frac{1}{2}(u_k^0)^{\mathrm{T}} M u_k^0 - \frac{1}{2}(x_k^0)^{\mathrm{T}} K x_k^0, \end{aligned} \tag{18}$$

where u_k^0 and x_k^0 are related by the discrete kinematics:

$$u_k^0 = \frac{x_{k+1}^0 - x_k^0}{h}. \tag{19}$$

Let $\mathcal{N}_0^k$ denote the set of sensors observing the target at time t_k. The following statement gives a discrete-time variational estimator, as a counterpart to the continuous-time variational estimator of Proposition 3.1, for the target tracking problem being considered here.

Proposition 4.1: Let $\mathcal{L}(x_k^0, u_k^0)$ be the discrete Lagrangian defined by (11), h be the discrete sampling period, and let $C \in \mathbb{R}^{3\times 3}$ be a positive definite matrix. Then the discrete variational estimator for the translational motion states of the target, obtained by applying the discrete Lagrange-d'Alembert principle to this Lagrangian with the dissipation term $F_{D_k} = -Cu_k^0$, is given by

$$\begin{aligned} \widehat{b}_{k+1}^0 &= \widehat{b}_k^0 + h\widehat{v}_k^0, \\ \widehat{v}_k^0 &= \sum_{i\in\mathcal{N}_0^k} \lambda_i(v_k^i - v_k^{0i}) + u_k^0, \\ (M + hC)u_{k+1}^0 &= Mu_k^0 - hKx_{k+1}^0, \end{aligned} \tag{20}$$

where x_k^0 and u_k^0 are related as in (19).

Proof: The first of eqs. (20) is the discrete kinematics equation relating the estimates $\widehat{b}_k^0$ and $\widehat{v}_k^0$, and the second equation is a consequence of eq. (7). The discrete version of the Lagrange-d'Alembert principle was obtained in seminal work on discrete geometric mechanics in the late 1990's and early 2000's [16], [19]. As a first step in obtaining the discrete Lagrange-d'Alembert equations for a mechanical system with non-conservative forcing, a discrete action sum is created from the discrete Lagrangian as follows:

$$\mathcal{S}_d(\mathcal{L}(x_k^0, u_k^0)) = \sum_{k=0}^{N} h\mathcal{L}(x_k^0, u_k^0) = \sum_{k=0}^{N} h\mathcal{L}_k. \tag{21}$$

The discrete Lagrange-d'Alembert principle states that with a "non-conservative forcing" F_{D_k} applied to the system given by the state estimation errors, the following is satisfied:

$$\delta\mathcal{S}_d(\mathcal{L}(x_k^0, u_k^0)) + h\sum_{k=0}^{N-1} F_{D_k}^{\mathrm{T}} \delta x_k^0 = 0, \tag{22}$$

where δq_k denotes the first *admissible variation* of the quantity q_k with fixed endpoints, i.e., $\delta q_0 = \delta q_N = 0$ [15], [20]. Applying (22) to the discrete action sum (21) with the dissipative forcing term $F_{D_k} = -Cu_k^0$ and the discrete kinematics (19), we get

$$\sum_{k=0}^{N-1} \Big[(u_k^0)^{\mathrm{T}} M(\delta x_{k+1}^0 - \delta x_k^0) - h(x_k^0)^{\mathrm{T}} K\delta x_k^0 - h(u_k^0)^{\mathrm{T}} C\delta x_k^0 \Big] = 0.$$

Collecting terms linearly dependent upon δx_k^0 which is an arbitrary admissible variation satisfying $\delta x_0^0 = \delta x_N^0 = 0$, we get

$$\Big[(u_{k-1}^0)^{\mathrm{T}} M - (u_k^0)^{\mathrm{T}} M - h(x_k^0)^{\mathrm{T}} K - h(u_k^0)^{\mathrm{T}} C \Big] \delta x_k^0 = 0 \ \text{for}\ k = 1, \ldots, N-1. \tag{23}$$

Finally, noting that the δx_k are arbitrary admissible variations, we conclude that the term in square brackets $[\ldots]$ vanishes. This gives rise to the last of eqs. (20). ■

Note that this discrete time variational estimator retains the property of dissipation of the total energy function (evaluated in discrete time), just like its continuous-time counterpart [12]. This leads to discrete-time stability of this discrete variational estimator.

B. Numerical Simulation Results

The discrete variational estimator given by Proposition 4.1 is numerically simulated here in the following form:

$$\begin{aligned} \widehat{b}^0_{k+1} &= \widehat{b}^0_k + h\widehat{v}^0_k, \\ \widehat{v}^0_k &= \sum_{i\in\mathcal{N}_0^k} \lambda_i(v^i_k - v^{0i}_{k,m}) + u^0_k, \\ (M+hC)u^0_{k+1} &= Mu^0_k - hKx^0_{k+1}, \end{aligned} \tag{24}$$

where $v^{0i}_{k,m}$ is the measured value of v^{0i}_k. Note that x^0_{k+1} is obtained from eq. (6) evaluated at time t_{k+1}, using the value of $\widehat{b}^0_{k+1}$ given by the first of eqs. (24). This value of x^0_{k+1} is then substituted in the last of eqs. (24) to get the update of the velocity estimation error, u^0_{k+1}. By utilizing MATLAB software, following results were generated. First we show how the target is moving; Fig. 2 shows its trajectory in a three dimensional space, Fig. 3 shows its position components and Fig. 4 shows components of the velocity of the target. We assume a scenario where three sensors can observe the target for 7.5 seconds and suddenly one of the sensors can not detect the target anymore at $t = 7.5s$, hence, the estimation should continue with only two sensors. We show the results of this scenario for two cases; first when there is no measurement noise in sensors and second, when the sensors have random noisy measurements. Comparison of the results show the robustness of the proposed algorithm when there is measurement noise.

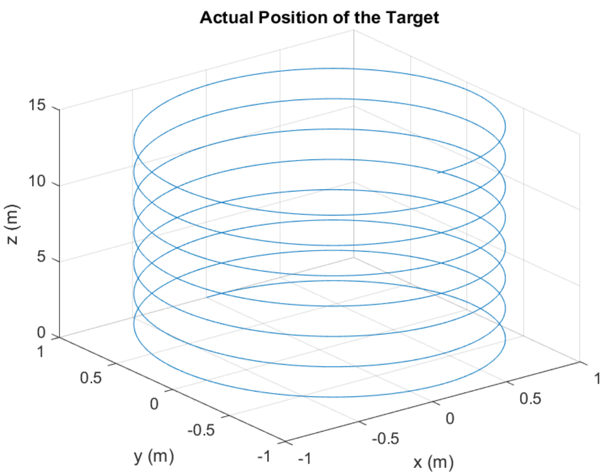

Fig. 2. Actual position b^0 of the target

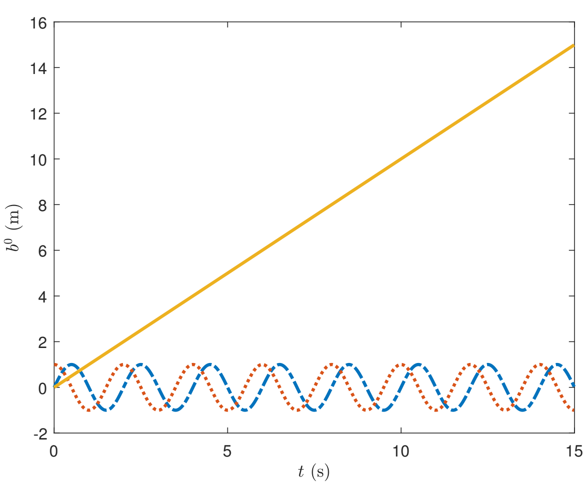

Fig. 3. Components of the position b^0 of the target

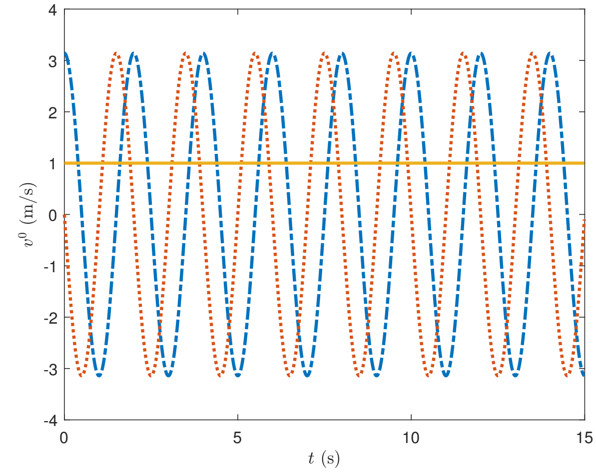

Fig. 4. Components of the velocity v^0 of the target

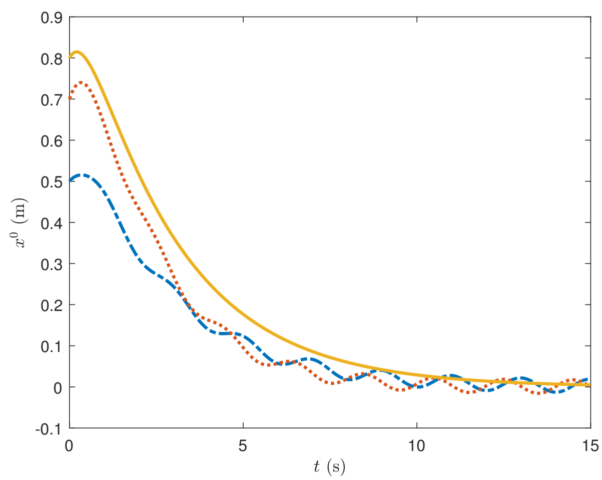

Fig. 5. Position Estimation Errors when there is no noise

Figures 5 and 6 show the behavior of position and velocity estimation errors, respectively, and how they converge to zero when there is no noise. Even when one of the sensors does not work at $t = 7.5s$ and onwards, the algorithm enforces asymptotic convergence of the errors to zero.

Figures 7 and 8 show the position estimation error and velocity estimation error over time, when there is sensor measurement noise. These two figures demonstrate how robust the proposed estimation scheme is to high frequency

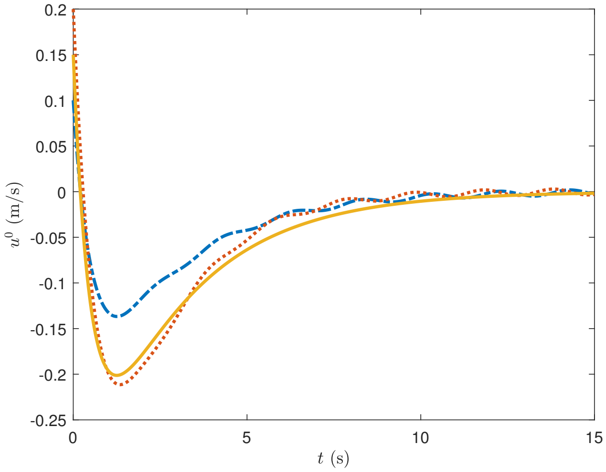

Fig. 6. Velocity Estimation Errors when there is no noise

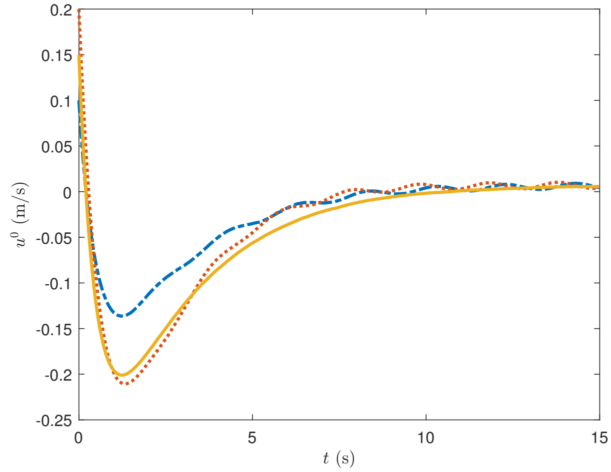

Fig. 8. Velocity Estimation Errors in presence of noise

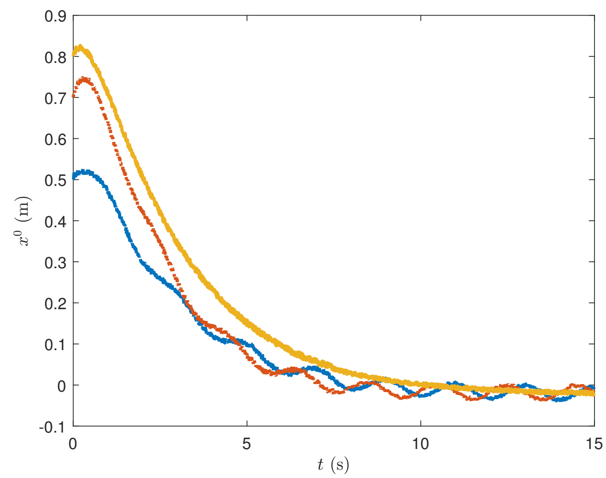

Fig. 7. Position Estimation Errors in presence of noise

measurement noise. In addition, comparing figure 7 with figure 5 and figure 8 with figure 6, we can observe how well the estimator works even in presence of sensor measurement noise.

5. Conclusion

In this article, we obtained a distributed observer design, in the form of a variational estimator, for estimating the translational motion states (position and velocity vectors) of a mobile target observed by a subset of sensors in a mobile sensor network. The mobile sensors are assumed to maintain a strongly connected communications network and communicate information about their states and measurements of target's relative motion among themselves. Measurements of the target by the sensors that directly observe it, are used to design a distributed observer in the form of a variational estimator. This distributed observer can be then implemented in a decentralized manner on each sensor. The variational estimator for the target's position and velocity vectors in three spatial dimensions is derived using the Lagrange-d'Alembert principle from variational mechanics, and shown (theoretically) to be asymptotically stable. A discrete version of this variational estimator is also obtained using the discrete Lagrange-d'Alembert principle, for ease of numerical implementation. Numerical simulations carried out using this discrete variational estimator confirm its stability properties and its robustness to high-frequency measurement noise.

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